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Rate of Convergence for the Modified Szász–Mirakyan Operators on Functions of Bounded Variation

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In this paper we obtain an estimate of the rate of convergence of modified Szász–Mirakyan operators on functions of bounded variation. Our result essentially improves the results due to A. Sahai and G. Prasad (1993, *Publ. Inst. Math.* (*Beograd*) (*N.S.*) **53**, 73–80) and V. Gupta and P. N. Agrawal (1991, *Publ. Inst. Math.* (*Beograd*) (*N.S.*) **49**, 97–103). © 1999 Academic Press

1. INTRODUCTION

The modified Szász–Mirakyan operator [4] is defined as

$$M_n(f,x) = n \sum_{k=0}^{\infty} p_k(nx) \int_0^{\infty} p_k(nt) f(t)\, dt, \qquad x \in [0,\infty), \quad (1.1)$$

where

$$p_k(nx) = e^{-nx}(nx)^k/k!.$$

Gupta and Agrawal [3] estimated the rate of convergence for the operator (1.1) for functions of bounded variation. Recently, Sahai and Prasad [5]

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improved and corrected the results of [3]. They have taken the function to be of growth $e^{\alpha t}$, $\alpha > 0$. The improved estimate obtained by Sahai and Prasad [5] is not entirely correct. The aim of this paper is to improve and correct the results of Sahai and Prasad [5] and Gupta and Agrawal [3]. The main result obtained by Sahai and Prasad [5] is

THEOREM A. *Let f be a function of bounded variation on every finite subinterval of $[0,\infty)$ and let $f(t) = O(e^{\alpha t})$ for some $\alpha > 0$ as $t \to \infty$. If $x \in (0,\infty)$, then for n sufficiently large*

$$\begin{aligned}\left|M_n(f,x) - \frac{1}{2}\{f(x+) + f(x-)\}\right| &\le \frac{(x^2+6x+3)x^{-2}}{n} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) \\ &\quad + \frac{(4x^2+3x+1)x^{-1/2}}{\sqrt{n}} |f(x+) - f(x-)| \\ &\quad + O(1)e^{-\alpha x}\frac{(2x+1)x^{-2}}{n},\end{aligned}$$

where $V_a^b(g_x)$ is the total variation of g_x on $[a,b]$ and

$$g_x \equiv g_x(t) = \begin{cases} f(t) - f(x+), & \text{if } x < t < \infty \\ 0, & \text{if } t = x \\ f(t) - f(x-), & \text{if } 0 \le t < x. \end{cases} \tag{1.2}$$

2. AUXILIARY RESULTS

To prove our main result, we shall need the following lemmas:

LEMMA 2.1. *For every $x \in (0,\infty)$, we have*

$$p_k(nx) \le \frac{\phi(x)}{\sqrt{n}}, \quad \text{where} \quad \phi(x) = \frac{32x^2 + 24x + 5}{2\sqrt{x}}.$$

Proof. Let $\{\xi_i\}$ be a sequence of independent and identically distributed random variables all having the same Poisson (x) distribution. Let $\eta_n = \sum_{k=1}^{n} \xi_k$; then

$$P(\eta_n = k) = e^{-nx}\frac{(nx)^k}{k!} = p_k(nx).$$

Also,

$$\rho_2 = x,\ \beta_3 = E|\xi_1 - x|^3 \le E\left(\xi_1^3\right) + 3xE\left(\xi_1^2\right) + 3x^2E(\xi_1) + x^3$$
$$= 8x^3 + 6x^2 + x.$$

Next,

$$p_k(nx) = P(k - 1 < \eta_n \le k) = P\left(\frac{k - 1 - nx}{\sqrt{nx}} < \frac{\eta_n - nx}{\sqrt{nx}} \le \frac{k - nx}{\sqrt{nx}}\right).$$

By [1, pp. 104 and 110; 2], we have

$$\left| p_k(nx) - \frac{1}{\sqrt{2\pi}} \int_{(k-1-nx)/\sqrt{nx}}^{(k-nx)/\sqrt{nx}} e^{-t^2/2}\, dt \right| < 2(0.82)\frac{8x^2 + 6x + 1}{\sqrt{nx}}$$
$$< \frac{16x^2 + 12x + 2}{\sqrt{nx}}.$$

Now

$$\frac{1}{\sqrt{2\pi}} \int_{(k-1-nx)/\sqrt{nx}}^{(k-nx)/\sqrt{nx}} e^{-t^2/2}\, dt < \frac{1}{\sqrt{2\pi nx}} < \frac{1}{2\sqrt{nx}}.$$

Therefore

$$p_k(nx) \le \frac{16x^2 + 12x + 2}{\sqrt{nx}} + \frac{1}{2\sqrt{nx}} = \frac{\phi(x)}{\sqrt{n}}.$$

Remark. We observe that the constant 0.41 taken in [5] holds only for sufficiently large n. For all n, we should take it 0.82 as given in [1, 2].

For $n \ge 2$, we have $2x/n \le M_n((t - x)^2, x) \le (2x + 1)/n$. If $K_n(x,t) = n\sum_{k=0}^{\infty} p_k(nx)p_k(nt)$, then it is easy to verify that

(i) for $0 \le y < x$, we have

$$\int_0^y K_n(x,t)\, dt \le \frac{2x + 1}{n(x - y)^2} \tag{2.1}$$

(ii) for $x < z < \infty$, we have

$$\int_z^\infty K_n(x,t)\, dt \le \frac{2x + 1}{n(z - x)^2}. \tag{2.2}$$

3. MAIN RESULT

THEOREM 3.1

. *Let f be a function of bounded variation on every finite subinterval of* $[0,\infty)$ *and let* $f(t) = O(e^{\alpha t})$ *for some* $\alpha > 0$ *as* $t \to \infty$. *If* $x \in (0,\infty)$ *and* $n \geq 4\alpha$, *then*

$$\left| M_n(f,x) - \frac{1}{2}\{f(x+) + f(x-)\} \right|$$
$$\leq \frac{(x^2 + 6x + 3)}{nx^2} \sum_{k=1}^{n} V_{x - x/\sqrt{k}}^{x + x/\sqrt{k}}(g_x)$$
$$+ \frac{(32x^2 + 24x + 5)}{2\sqrt{nx}} |f(x+) - f(x-)|$$
$$+ \sqrt{\frac{2(2x+1)}{n}} \frac{e^{2\alpha x}}{x} + \frac{e^{\alpha x}(2x+1)}{nx^2}, \tag{3.1}$$

where $V_a^b(g_x)$ *is the total variation of* g_x *on* $[a,b]$ *as given in* (1.2).

Proof. First we have

$$\left|M_n(f,x) - \tfrac{1}{2}\{f(x+) + f(x-)\}\right|$$
$$\leq |M_n(g_x, x)| + \tfrac{1}{2}|f(x+) - f(x-)| \cdot |M_n(\text{sign}(t-x), x)|. \tag{3.2}$$

Thus, to estimate the left hand side, we need estimates for $M_n(g_x, x)$ and $M_n(\text{sign}(t-x), x)$. We have

$$\begin{aligned} M_n(\text{sign}(t-x), x) &= \int_0^\infty \text{sign}(t-x) K_n(x,t)\, dt \\ &= \int_x^\infty K_n(x,t)\, dt - \int_0^x K_n(x,t)\, dt \\ &= A_n(x) - B_n(x), \quad \text{say}. \end{aligned}$$

Proceeding as in the proof of the theorem in [5], we have

$$|A_n(x) - B_n(x)| \leq \frac{\phi(x)}{\sqrt{n}}.$$

To estimate $M_n(g_x, x)$, we decompose $[0, \infty)$ into three parts, as follows:

$$\begin{aligned} M_n(g_x, x) &= \int_0^{\infty} K_n(x,t) g_x(t)\, dt \\ &= \left(\int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} + \int_{x+x/\sqrt{n}}^{\infty} \right) K_n(x,t) g_x(t)\, dt \\ &= \left(\int_{I_1} + \int_{I_2} + \int_{I_3} \right) K_n(x,t) g_x(t)\, dt = E_1 + E_2 + E_3, \quad \text{say.} \end{aligned}$$

First we estimate E_2. For $t \in I_2$, we have

$$|g_x(t)| = |g_x(t) - g_x(x)| \le V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x),$$

and so

$$E_2 \le V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} K_n(x,t)\, dt = V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} d_t \lambda_n(x,t),$$

where

$$\lambda_n(x,t) = \int_0^t K_n(x,u)\, du.$$

Since $\int_a^b d_t \lambda_n(x,t) \le 1$ for all $[a,b] \subseteq [0, \infty)$, we have

$$E_2 \le V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \le \frac{1}{n} \sum_{k=0}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x). \tag{3.3}$$

Next, using (2.1) and proceeding as in [5], we have

$$E_1 \le \frac{2(2x+1)}{nx^2} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x}(g_x). \tag{3.4}$$

Finally, we estimate E_3. Setting $z = x + x/\sqrt{n}$, we obtain

$$E_3 = \int_z^{\infty} g_x(t) K_n(x,t)\, dt = \int_z^{\infty} g_x(t)\, d_t(\lambda_n(x,t)).$$

We define $Q_n(x,t)$ on $[0, 2x]$ as

$$Q_n(x,t) = \begin{cases} 1 - \lambda_n(x, t-), & \text{if } 0 \le t < 2x \\ 0, & \text{if } t = 2x. \end{cases}$$

Therefore

$$E_3 = \int_z^{2x} g_x(t)\, d_t(Q_n(x,t)) - g_x(2x)\int_{2x}^{\infty} K_n(x,u)\, du$$

$$+ \int_{2x}^{\infty} g_x(t)\, d_t(\lambda_n(x,t))$$

$$= E_{31} + E_{32} + E_{33}, \quad \text{say.} \tag{3.5}$$

Now using (2.2) and proceeding as in [5], we have

$$|E_{31}| \le \frac{2(2x+1)}{nx^2} \sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x) \tag{3.6}$$

and

$$|E_{32}| \le \frac{(2x+1)}{nx^2} \sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x). \tag{3.7}$$

Finally, we estimate E_{33} as follows:

$$|E_{33}| \equiv \left| n \sum_{k=0}^{\infty} p_k(nx) \int_{2x}^{\infty} p_k(nt) g_x(t)\, dt \right|$$

$$\le n \sum_{k=0}^{\infty} p_k(nx) \int_{2x}^{\infty} p_k(nt)(e^{\alpha t} + e^{\alpha x})\, dt$$

$$= \frac{n}{x} \sum_{k=0}^{\infty} p_k(nx) \int_{2x}^{\infty} p_k(nt) x e^{\alpha t}\, dt + \frac{e^{\alpha x}}{x^2} n \sum_{k=0}^{\infty} p_k(nx) \int_{2x}^{\infty} p_k(nt) x^2\, dt$$

$$\le \frac{n}{x} \sum_{k=0}^{\infty} p_k(nx) \int_{2x}^{\infty} |t-x| e^{\alpha t} p_k(nt)\, dt$$

$$+ \frac{e^{\alpha x}}{x^2} n \sum_{k=0}^{\infty} p_k(nx) \int_{2x}^{\infty} p_k(nt) \cdot (t-x)^2\, dt$$

$$(\text{because for } t \ge 2x, \quad t - x \ge x)$$

$$\le \frac{n}{x} \sum_{k=0}^{\infty} p_k(nx) \int_{0}^{\infty} |t-x| e^{\alpha t} p_k(nt)\, dt$$

$$+ \frac{e^{\alpha x}}{x^2} n \sum_{k=0}^{\infty} p_k(nx) \int_{0}^{\infty} p_k(nt) \cdot (t-x)^2\, dt$$

$$\le \frac{1}{x} \left(n \sum_{k=0}^{\infty} p_k(nx) \int_0^{\infty} p_k(nt)(t-x)^2\, dt \right)^{1/2}$$

$$\times\left(n\sum_{k=0}^{\infty}p_k(nx)\int_0^{\infty}p_k(nt)\cdot e^{2\alpha t}\,dt\right)^{1/2}+\frac{e^{\alpha x}}{x^2}M_n\left((t-x)^2,x\right)$$

$$\leq\frac{1}{x}\left(M_n\left((t-x)^2,x\right)\right)^{1/2}\left(n\sum_{k=0}^{\infty}p_k(nx)\int_0^{\infty}p_k(nt)e^{2\alpha t}\,dt\right)^{1/2}$$

$$+\frac{e^{\alpha x}}{x^2}M_n\left((t-x)^2,x\right).$$

Next, we have by partial integration and by the assumption that $n > 2\alpha$

$$n\sum_{k=0}^{\infty}p_k(nx)\int_0^{\infty}p_k(nt)e^{2\alpha t}\,dt$$

$$=n\sum_{k=0}^{\infty}p_k(nx)\frac{n^k}{k!}\int_0^{\infty}t^k e^{-(n-2\alpha)t}\,dt$$

$$=n\sum_{k=0}^{\infty}p_k(nx)\frac{n^k}{k!}\frac{k!}{(n-2\alpha)^{k+1}}=\frac{n}{n-2\alpha}\sum_{k=0}^{\infty}\left(\frac{n}{n-2\alpha}\right)^k p_k(nx)$$

$$=\frac{n}{n-2\alpha}e^{-nx}\sum_{k=0}^{\infty}\left(\frac{n^2x}{(n-2\alpha)}\right)^k\frac{1}{k!}$$

$$=\frac{n}{n-2\alpha}\cdot e^{-nx}\cdot e^{nx(1+2\alpha/(n-2\alpha))}$$

$$=\frac{n}{n-2\alpha}\cdot e^{2\alpha nx/(n-2\alpha)}=\frac{n}{n-2\alpha}\cdot e^{2\alpha x\cdot n/(n-2\alpha)}$$

$$\leq 2e^{4\alpha x},\quad\text{for}\quad n\geq 4\alpha.$$

Therefore

$$|E_{33}|\leq\frac{1}{x}\left(\frac{2x+1}{n}\right)^{1/2}\cdot\left(2e^{4\alpha x}\right)^{1/2}+\frac{e^{\alpha x}}{x^2}\frac{2x+1}{n}$$

$$=\sqrt{\frac{2(2x+1)}{n}}\,\frac{e^{2\alpha x}}{x}+\frac{e^{\alpha x}}{x^2}\frac{2x+1}{n}. \tag{3.8}$$

Using (3.5) to (3.8), we have, for $n \geq 4\alpha$,

$$|E_3|\leq\frac{3(2x+1)}{nx^2}\sum_{k=1}^{n}V_x^{x+x/\sqrt{k}}(g_x)+\sqrt{\frac{2(2x+1)}{n}}\,\frac{e^{2\alpha x}}{x}$$

$$+\frac{e^{\alpha x}(2x+1)}{nx^2}. \tag{3.9}$$

Combining (3.2) with (3.3), (3.4) and (3.9), we get the required result.

Remark. We may remark that the estimate (3.9) in [5] seems to be incorrect. In the estimation of E_{33} in [5], the term $\{1 + \alpha/m\}^{2k+1}$ cannot be considered constant.

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